A Theoretical Steady State Profile of Ice Sheets (Two-Dimensional Model)

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定常氷床の理論的断面（2 次元モデル）

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要旨： 氷床の断面形状に関する Nye の理論を，氷床内の氷質量保存の原則に基づいて改訂した．Nye の氷床模型，さらにそれを改変した Haefeli の氷床模型の双方とも，質量保存則を満足せず，したがって両氷床模型とも定常状態を保ち得ないからである．

新たに提案する理論においては，氷床中の氷質量保存法則を

$$\int_0^s (b\cos\theta - u\sin\theta)\,ds = uh$$

で表し，Nye および Haefeli の仮定 $bx=uh$ とは異なる．ここに，$\cos\theta\cdot ds=dx$, $-\dfrac{dh}{dx}=\tan\theta$，また b と $h(x)$ はそれぞれ氷集積速度および氷床面の高さを表す．理論的計算の結果，定常氷床断面形状は

$$\left[1+\frac{m}{m+1}\left(\frac{h}{H}\right)\right]\left(1-\frac{h}{H}\right)^{\frac{m}{m+1}}=\frac{x}{L}$$

で与えられる（図 2)．ここに，$x=0$ で $h=H$，また $x=L$ で $h=0$ を以って H および L を定義してある．

この新定常氷床模型は，もちろん氷床内の氷質量保存の条件を満足するし，積氷域と剥氷域の境に当たる定常点を量的に定め得るほか，氷床内の氷流の流線分布（図 3）を求めることができる．

現在，もっとも詳しく測定されているグリーンランドの氷床断面と新定常氷床模型を比べた結果，両者の一致は十分満足できると結論できる（図 4)．

Abstract: The Nye's kinematic theory of ice sheet profiles is revised on the basis of the conservation law of ice mass, because the Nye's ice sheet model or the revised Haefeli's model does not satisfy the ice mass conservation law and therefore they cannot have a steady state.

In the present revised theory, the mass conservation condition is represented by

$$\int_0^s (b\cos\theta - u\sin\theta)\,ds = uh,$$

in place of Nye-Haefeli assumption that $bx=uh$, where $\cos\theta\cdot ds=dx$, $-\dfrac{dh}{dx}=\tan\theta$, and b and $h(x)$ denote respectively the ice accumulation rate and the height of ice sheet surface. Then, the revised ice sheet profile is expressed

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by

$$\left[1+\frac{m}{m+1}\left(\frac{h}{H}\right)\right]\left(1-\frac{h}{H}\right)^{\frac{m}{m+1}}=\frac{x}{L},$$

where $h=H$ at $x=0$ and $x=L$ at $h=0$. This ice sheet profile model satisfies the ice mass conservation within the ice sheet. In this model, the equilibrium point to separate the ablation area from the accumulation area is clearly defined, and the stream-lines of ice flow within the ice sheet are explicitly determined.

It seems further that the agreement of the present steady state theoretical model of ice sheet profile with the observed data in Greenland is satisfactory.

1. Introduction

A theoretical profile of the ice sheets proposed by NYE (1959) on the basis of a simplified kinematic model seems to be in reasonably good agreement with the observed profile of ice sheet (*e.g.* PATERSON, 1969). However, a significant difficulty in the Nye's theory is that the velocity of ice flow becomes infinite at the outer edge of an ice sheet so that the real steady state of an ice sheet cannot be present in this model.

In the Nye's two-dimensional kinematic model, where the total width and the thickness of an ice sheet are represented by $2L$ and h respectively in the x-z rectangular coordinates of horizontal x-axis and vertical z-axis, it is assumed that the amount of ice accumulating on the surface between the crest (at $x=0$) and any point P on the ice sheet surface is equal to the amount of ice flowing outwards through a vertical section at P (see Fig. 1.). When a uniform thickness of ice, b, is continuously added to a unit area of the surface per unit time, the horizontal velocity of ice flow, u, through the vertical section at P is given by

$$u=\frac{bx}{h}. \tag{1}$$

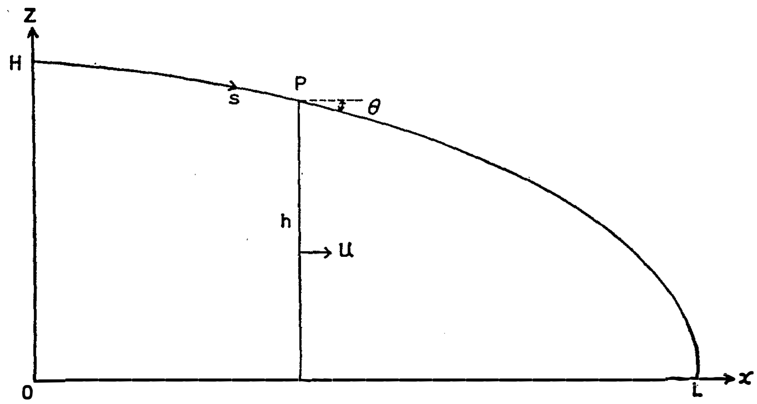

Fig. 1. Coordinates for ice sheet profile model.

Obviously, u becomes infinitely large at $x=L$ where $h=0$. As mathematically shown later, the mass balance of ice between the inflow caused by adding ice of b in rate and the outflow caused by the horizontal movement with u in velocity, both through the ice sheet surface, becomes infinitely large in the Nye's model. This result would mean that a real steady state of an ice sheet profile cannot be present in this model.

It is obvious that the mass conservation law must be satisfied even in such a simple kinematic theory of the ice sheet profile. In the present trial, it will be assumed that the accumulation process of ice is exactly the same as in the case of Nye's model, but the outward flow of ice through the ice sheet surface is taken off into the outer atmosphere by the ablation effect from the ice sheet mass. Taking into consideration the mass conservation law in an explicit way as mentioned, the Nye's simple theory of ice sheet profile will be reasonably well modified to satisfy all the necessary conditions for a steady state model.

2. Modified Profile of Ice Sheet

All basic assumptions in the present theoretical trial are same as those adopted in the Nye's theory, except one condition that the ablation effect on the ice sheet surface is explicitly taken into account.

In accordance with the notations of s and θ given in Fig. 1, the accumulation rate of ice on ds is $b\cos\theta\cdot ds$, while the ablation rate on ds is represented by $u\sin\theta\cdot ds$. Thus, the net accumulation rate dv per ds is given by

$$dv=(b\cos\theta-u\sin\theta)\,ds. \qquad (2)$$

It may be clear that dv represents the vertically inward flow of ice through ds and it must be considered that dv is taken out into the outer atmosphere by some ablation effects when $dv<0$, provided that the ice sheet profile expressed by $h(x)$ represents its steady state.

Then, the mass conservation law can be expressed by

$$\int_0^s (b\cos\theta-u\sin\theta)\,ds=hu, \qquad (3)$$

or

$$\int_0^x \left(b+u\frac{dh}{dx}\right)dx=hu, \qquad (4)$$

and eq. (4) leads to

$$\frac{du}{dx}=\frac{b}{h(x)}, \qquad (5)$$

where b is assumed to be constant.

On the other hand, the stress and strain rate law on the bottom surface of ice sheet may be represented, according to Nye's original expression, as

$$u = A'\left(-\rho gh\frac{dh}{dx}\right)^m, \tag{6}$$

where ρ, g and A' denote respectively the density of ice, the gravity accerelation and a material constant. Eq. (6) can be rewritten as

$$h\frac{dh}{dx} = -\frac{u^{\frac{1}{m}}}{\beta}, \tag{7}$$

where $\beta \equiv \rho g A'^{\frac{1}{m}}$. Combining eq. (5) with eq. (7), we get

$$u^{\frac{1}{m}}du = -b\,\beta\,dh. \tag{8}$$

Integrating eq. (8) and putting $u=0$ and $h=H$ at $x=0$, we get

$$\left(\frac{m}{m+1}\right)u^{\frac{m+1}{m}} = b\,\beta(H-h). \tag{9}$$

Then, combining eq. (7) and eq. (9), we get a $h \sim x$ relation as

$$\frac{h\,dh}{(H-h)^{\frac{1}{m+1}}} = -\frac{\left(\frac{m+1}{m}b\beta\right)^{\frac{1}{m+1}}}{\beta}\,dx. \tag{10}$$

The integration of eq. (10) with a boundary condition of $h=0$ at $x=L$ is given by

$$\frac{(m+1)^2}{m(2m+1)}H^{\frac{2m+1}{m+1}}\left[1-\left(1+\frac{m}{m+1}\cdot\frac{h}{H}\right)\left(1-\frac{h}{H}\right)^{\frac{m}{m+1}}\right] = \left(\frac{m+1}{m}\,b\right)^{\frac{1}{m+1}}\beta^{-\frac{m}{m+1}}L\left(1-\frac{x}{L}\right). \tag{11}$$

Then, considering that $h=H$ at $x=0$, we can rewrite eq. (11) as

$$\left(1+\frac{m}{m+1}\cdot\frac{h}{H}\right)\left(1-\frac{h}{H}\right)^{\frac{m}{m+1}} = \frac{x}{L}, \tag{12}$$

with

$$\frac{(m+1)^2}{m(2m+1)}H^{\frac{2m+1}{m+1}} = \left(\frac{m+1}{m}\,b\right)^{\frac{1}{m+1}}\beta^{-\frac{m}{m+1}}L. \tag{13}$$

As well known, the Nye's ice sheet profile is expressed as

$$\left(\frac{h}{H}\right)^{\frac{2m+1}{m}} + \left(\frac{x}{L}\right)^{\frac{m+1}{m}} = 1, \tag{14}$$

with

$$\frac{m+1}{2m+1} H^{\frac{2m+1}{m}} = b^{\frac{1}{m}} \beta^{-1} L^{\frac{m+1}{m}}. \tag{15}$$

As generally discussed by PATERSON (1969), the observed results of ice sheet profile indicate that $m=2\sim3$ and most likely $m\simeq2$. Fig. 2 illustrates the present revised profile of ice sheet given by eq. (12) together with the Nye's profile expressed by eq. (14) for the case of $m=2$. As shown in the figure, the revised profile is not too much different from the Nye's one, though the apparent mathematical expression given by eq. (12) looks considerably different from that of eq. (14).

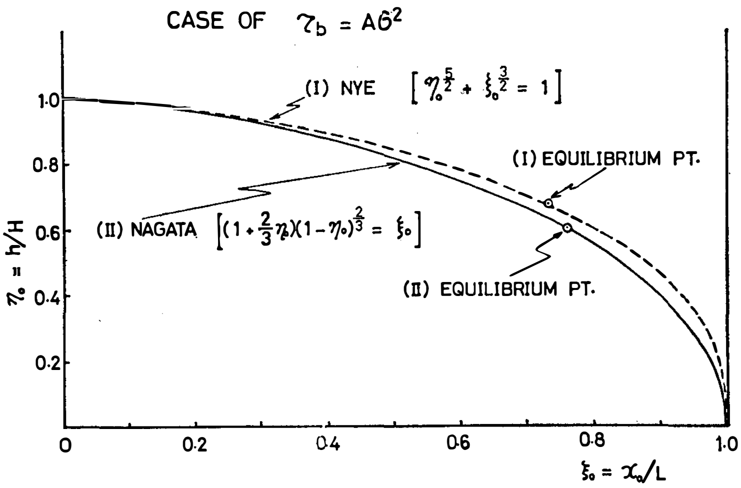

Fig. 2. The theoretical steady state profile of an ice sheet in comparison with the Nye's model for $m=2$.

However, the relative relation between H and L for the same physical parameters such as β and b is different in eq. (13) from that in eq. (15), because eq. (13) can be rewriten as

$$\frac{m+1}{2m+1} H^{\frac{2m+1}{m}} = \left(\frac{m}{m+1}\right) \left(\frac{2m+1}{m+1}\right)^{\frac{1}{m}} b^{\frac{1}{m}} \beta^{-1} L^{\frac{m+1}{m}}. \tag{13'}$$

Since $\left(\frac{m}{m+1}\right) \left(\frac{2m+1}{m+1}\right)^{\frac{1}{m}} < 1$ for $m>0$, the H value in the present revised model is always smaller than that in the Nye's model as far as the L value is fixed.

3. Ice Mass Conservation

A steady state of the ice sheet profile can be maintained only if the ice inflow through the accumulation area surface is balanced with the ice outflow through the ablation area surface. The total net accumulation rate over the whole ice sheet surface is given from eq. (3) as

$$I(L)=\int_0^L (b\cos\theta-u\sin\theta)\frac{ds}{dx}\,dx$$

$$=bL-\int_0^H u\,dh. \tag{16}$$

The integration of eq. (16) by considering eq. (9) leads to

$$I(L)=bL-\left(\frac{m+1}{2m+1}\right)\left(\frac{m+1}{m}\,b\,\beta\right)^{\frac{m}{m+1}} H^{\frac{2m+1}{m+1}}. \tag{17}$$

Putting eq. (13) into eq. (17), we find that

$$I(L)=0.$$

Namely, the ice mass conservation law holds in the present ice sheet profile model. Actually, the velocity (u_o) of ice outflow at the outer edge of ice sheet (at $x=L$, $h=0$) takes a finite value and is given by eq. (9) as

$$u_o^{\frac{m+1}{m}}=\frac{m+1}{m}\,b\,\beta\,H. \tag{18}$$

In comparison, the u_o value in the Nye's model is infinitely large, and $I(L)$ also becomes infinitely large, because

$$I_N(L_N)=bL_N-bL_N\int_0^H\left[1-\left(\frac{h}{H_N}\right)^{\frac{2m+1}{m}}\right]^{\frac{m}{m+1}}\frac{dh}{h}, \tag{16'}$$

where $I_N(L_N)$ represents the mathematical operation expressed by eq. (16) in the case of Nye's model. This result may indicate that the Nye's ice sheet model cannot have a steady state and the ice sheet must disappear with time in spite of a continuous ice accumulation represented by b.

In the present model, an area of $b\cos\theta-u\sin\theta>0$ represents the accumulation area, while that of $b\cos\theta-u\sin\theta<0$ the ablation area, the equilibrium point being determined by

$$b\cos\theta-u\sin\theta=\left(b+u\frac{dh}{dx}\right)\cos\theta=0. \tag{19}$$

Putting eq. (9), eq. (12) and eq. (13) into eq. (19), we can get

$$\frac{h_e}{H}=\frac{m+1}{2m+1},\quad \frac{x_e}{L}=\left(\frac{3m+1}{2m+1}\right)\left(\frac{m}{2m+1}\right)^{\frac{m}{m+1}}, \tag{20}$$

where h_e and x_e denote respectively h and x values at the equilibrium point. Thus, the accumulation area can be defined by $x/L<x_e/L$ or $h/H>h_e/H$, whereas the ablation area by $x/L>x_e/L$ or $h/H<h_e/H$. This conclusion is

obvious from a caluculated result that

$$\int_0^{x_e}(b-u\tan\theta)\,dx = bL\left(\frac{m}{2m+1}\right)^{\frac{m}{m+1}} = -\int_{x_e}^{L}(b-u\tan\theta)\,dx. \tag{21}$$

In the case of $m=2$ which may represent the most realistic ice sheet profile,

$$h_e = \frac{3}{5}H \text{ and } x_e = \left(\frac{7}{5}\right)\left(\frac{2}{5}\right)^{\frac{2}{3}} L \simeq 0.760\,L.$$

In the Nye's ice sheet profile also, the equilibrium point may be defined by the point where $b+u\frac{dh}{dx}=0$. This condition is satisfied by

$$\left(\frac{h_e}{H}\right)^{\frac{2m+1}{m}}\left(\frac{x_e}{L}\right)^{\frac{m+1}{m}} = \frac{m+1}{2m+1}. \tag{22}$$

Then, putting eq. (14) into eq. (22), we get

$$\frac{x_e}{L} = \left(\frac{2m+1}{3m+1}\right)^{\frac{m}{m+1}},\quad \frac{h_e}{H} = \left(\frac{m}{3m+1}\right)^{\frac{m}{2m+1}}, \tag{23}$$

for this case. The equilibrium point thus defined for both the Nye's model and the present modified one is illustrated in Fig. 2 for the case of $m=2$.

4. Stream Lines of Ice Flow within the Ice Sheet

When the horizontal and vertical velocities of ice flow with in an ice sheet are noted by u and w respectively, the stream line of ice flow can be represented by

$$\frac{dx}{u} = \frac{dz}{w}. \tag{24}$$

Since the equation of continuity is expressed by

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

and $\frac{du}{dx}$ given by eq. (5) is independent of z, the vertical velocity, w, on the condition that $(w)_{z=0}=0$ and $(w)_{z=h}=-b$ must be given by

$$w(x) = -\frac{b}{h}z. \tag{25}$$

On the other hand, combining eq. (9) with eq. (12) and eq. (13), we get

$$u=\left(\frac{2m+1}{m+1}\right)\frac{bx}{H\left(1+\frac{m}{m+1}\frac{h}{H}\right)},$$

whence eq. (24) becomes

$$\left(\frac{m+1}{2m+1}\right)H\frac{dx}{xh}+\left(\frac{m}{2m+1}\right)\frac{dx}{x}+\frac{dz}{z}=0,$$

and therefore

$$\left(\frac{m+1}{2m+1}\right)H\int\frac{dx}{xh}+\frac{m}{2m+1}\log x+\log z=\text{constant}. \qquad (26)$$

The first term of the left-hand side of eq. (26) can be integrated with the aid of eq. (12), thus eq. (26) becoming

$$\left(\frac{m}{2m+1}\right)\log\left(\frac{1-\frac{h}{H}}{1+\frac{m}{m+1}\cdot\frac{h}{H}}\right)+\frac{m}{2m+1}\log x+\log z=\text{constant}. \qquad (26')$$

Noting $\frac{x}{L}\equiv\xi$, $\frac{h}{H}\equiv\eta_0$, and $\frac{z}{H}\equiv\zeta$, eq. (26′) can be rewriten as

$$\left(\frac{1-\eta_0}{1+\frac{m}{m+1}\eta_0}\right)^{\frac{m}{2m+1}}\xi^{\frac{m}{2m+1}}\zeta=\text{constant}, \qquad (27)$$

or, with the aid of eq. (12),

$$\xi\zeta\left(1+\frac{m}{m+1}\eta_0\right)^{-\frac{m+1}{2m+1}}=\text{constant}, \qquad (28)$$

where $\eta_0(\xi)$ is a function of ξ and expressed by

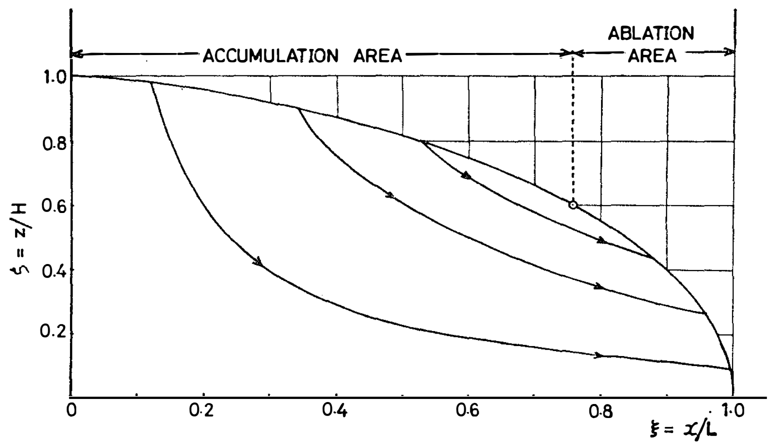

Fig. 3. Stream lines of ice flow within a steady state ice sheet profile. ($m=2$)

$$\left(1+\frac{m}{m+1}\eta_o\right)(1-\eta_o)^{\frac{m}{m+1}}=\xi. \tag{13'}$$

An example of the stream lines of ice flow thus obtained is shown in Fig. 3 for the case of $m=2$. It will be clearly seen in the figure that any ice mass accumulating on the accumulation area surface flows downwards along a stream line and comes out from the ablation area surface and that the equilibrium point represents the lowest limit in height where the ice mass can move into the ice sheet.

In the Nye's model of ice sheet profile, the stream lines of ice flow can be determined in a similar way. Namely,

$$\frac{dx}{\frac{bx}{h}}=-\frac{dz}{\frac{bz}{h}}$$

for the Nye's model, and consequently the stream lines are expressed by $xy=$ constant or

$$\xi\,\zeta = \text{constant}. \tag{29}$$

In this case also, any ice mass accumulating on the accumulation area surface flows down along a stream line and comes out from the ablation area surface.

5. Comparison of the Present Steady State Model with the Nye's Model

Although the Nye's model of ice sheet profile includes the essential difficulty with respect to the mass conservation, the shapes of the ice sheet profile and the stream lines derived from this model are not essentially different from those derived from the present self-consistent model of a steady state.

When m becomes very large, the Nye's ice sheet profile approaches to a parabolic form as expressed by

$$\eta_0^2 + \xi_0 = 1,\ (\text{for } m \to \infty), \tag{30}$$

whereas the present self-consistent steady state profile also approaches to the same expression as given by eq. (30). The geometrical characteristics of stream lines of ice flow in the Nye's model are not qualitatively different from those in the present self-consistent steady model. Namely, all ice masses accumulating on the accumulation area come out from the ablation area surface, and the position of equilibrium point to separate the ablation area from the accumulation one is not significantly different from that derived from the present theory.

As already discussed in Section 3, significant difficulty in the Nye's theory is concerned with the horizontal velocity of ice flow which becomes infinitely large with x approaching to L. If we do not deal with the ice flow velocity, therefore, the Nye's model might be considered as a simple but reasonably good approximation of an ice sheet profile model. The essential difference of the Nye's theory from the present one is based only on a rough approximation that h is assumed to be independent of x in eq. (5). If h assumes to be constant in eq. (5), u can be represented by eq. (1).

The reason why importance is still put on the Nye's classical model is due to its extremely simple concept represented by eq. (1), because the Nye's two-dimensional model can be directly extended to a three-dimensional case (*e.g.* HAEFELI, 1961). If, for instance, we consider a pancake shape ice sheet having a circular symmetricity, eq. (1) can be replaced in a cylindrical coordinate system, (r, θ, z), by

$$\pi r^2 b = 2\pi r hu,$$

or

$$u = \frac{rb}{2h}. \tag{31}$$

In an exactly same way as in the Nye's theoretical approach, eq. (6) together with eq. (31) leads to a conclusion that

$$\left(\frac{h}{H}\right)^{\frac{2m+1}{m}} + \left(\frac{r}{R}\right)^{\frac{m+1}{m}} = 1, \tag{32}$$

where $h=0$ at $r=R$ and $h=H$ at $r=0$ are assumed. The mathematical form of eq. (32) is exactly the same as that of eq. (14), and the relationship between H and R is given by

$$\frac{m+1}{2m+1} H^{\frac{2m+1}{m}} = b^{\frac{1}{m}} R^{\frac{m+1}{m}} / \beta', \tag{33}$$

which is of the same mathematical form as eq. (15), but

$$\beta' \equiv \rho g (2A')^{\frac{1}{m}}.$$

The equilibrium circle and the stream lines of ice flow in this three-dimensional ice sheet model can be obtained as

$$\frac{r_e}{R} = \left(\frac{4m+2}{5m+3}\right)^{\frac{m}{m+1}}, \quad \frac{h_e}{H} = \left(\frac{m+1}{5m+3}\right)^{\frac{m}{2m+1}}, \tag{34}$$

and

$$\left(\frac{r}{R}\right) \zeta^2 = \text{constant for stream lines.} \tag{35}$$

It will be certain that the shape of ice sheet given by eq. (32), the position of equilibrium line by eq. (34) and the stream lines by eq. (35) are quantitatively different from their respective corresponding values which could be derived from the self-consistent steady state model. Nevertheless, general characteristics of those parameters of ice sheet profiles may be qualitatively represented by these mathematical formulae. If we try to extend the present two-dimensional model of steady state ice sheet to a three-dimensional case, we must face an extreme complexity of mathematical expressions.

Recently effects of the bed rock topography upon the ice flow pattern within a three-dimensional ice sheet is becoming one of important problems specifically in connection with the transportation mechanism of meteorites and other solid materials by the ice flow (*e.g.* NAGATA *et al.*, 1975). It seems in this sense that the Nye's simplest possible model of ice sheet kinematics is still important as the basis for dealing with a very slow motion of ice sheet interior on various complicated boundary conditions.

It must be again emphasized, however, that any model of ice sheet based on the Nye's concept cannot quantitatively deal with the velocity of ice flow and the ice mass balance. In brief, the Nye's model could be considered as a topological model; it is not a dynamic model nor even a kinematic one, but it may be still useful for the purpose of a topological consideration of the ice sheet behavior.

6. Comparison of the Theoretical Steady State Ice Sheet Profile with Observed Data

Comparisons of the theoretical ice sheet profiles with the observed data in Antarctica and Greenland have been reported (*e. g.* HAEFELI, 1961; PATERSON, 1969). It has been generally accepted that the Nye's theoretical profile can reasonably well fit the observed profile data except for the neighborhood of the outer edge of an ice sheet.

One of studies in detail on this problem is a theoretical and analytical work by HAEFELI (1961) on the Greenland ice sheet. HAEFELI did not directly applied the Nye's theoretical profile in his analysis, but proposed a little modified theoretical model. In his case, the basic formula to represent the ice creep in an ice sheet is given by

$$\frac{\partial u}{\partial z} = k\tau^n,$$

instead of the Nye's basic assumption that

$$u = A' \tau_b^m$$

at the bottom of ice sheet and u is constant at all levels in an ice sheet along a line of x=constant. However, HAEFELI also made an assumption on the ice mass conservation that

$$bx=\int_0^h udz=hu_m. \qquad (36)$$

Eq. (36) is essentially the same as eq. (1) in the Nye's theory. On these assumptions, HAEFELI has derived an expression of an ice sheet profile as

$$\left(\frac{h}{H}\right)^{\frac{2(n+1)}{n}}+\left(\frac{x}{L}\right)^{\frac{n+1}{n}}=1. \qquad (37)$$

It may be obvious that the Haefeli's model also cannot have the ice mass conservation within an ice sheet and consequently the real steady state of ice sheet shape cannot be represented by this model. In his analysis of the Greenland ice sheet with eq. (37), HAEFELI picked up the crest point A (h=3,160 m, x=0) and point C (h=2,000 m, x=385 km) as the two reference points (see Fig. 4), and parameter n of the theoretical profile eq. (37) to fit the observed one was determined. It appears in his result that n=4 is the best, as shown in Fig. 4. The present steady state model is applied on the same data of Greenland ice sheet profile by taking A and C as the reference points. For the purpose of comparing the present steady state model with the Haefeli one, special points, P_1, P_2, P_3 and P_4 between A and C in his work are ex-

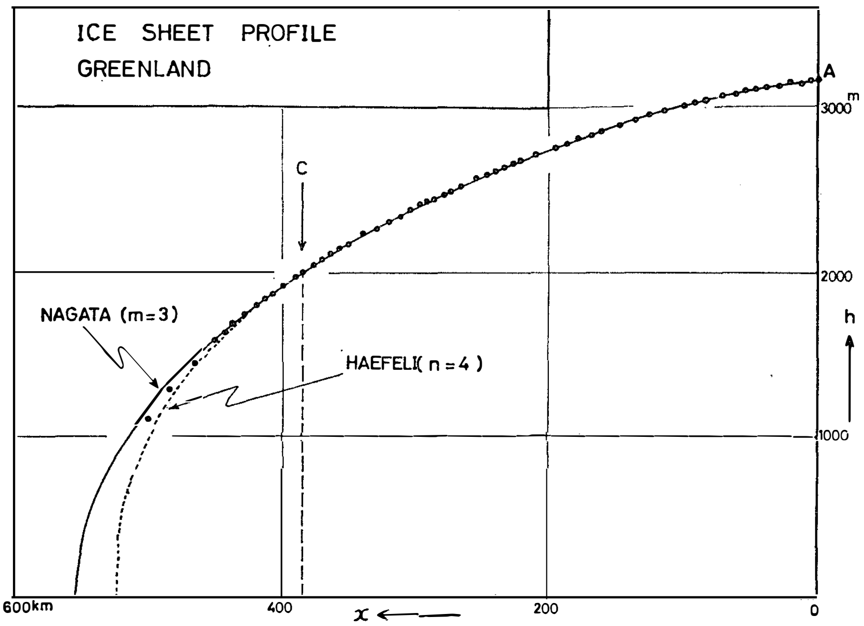

Fig. 4. Comparisons of the theoretical steady state model of an ice sheet profile (m=3) and the Haefeli's model (n=4) with the observed ice sheet profile of Greenland represented by full circles.

Table 1. Theoretical ice sheet profiles to approximate the Greenland ice sheet.

Points	h	Theoretical values of x		
		NAGATA (m=2)	NAGATA (m=3)	HAEFELI (n=4)
A	3,160m	(0)km	(0)km	(0)km
P_1	3,043	96.3	80.5	76.3
P_2	2,865	174.4	157.1	154.3
P_3	2,644	245.8	231.4	230.9
P_4	2,362	316	307.8	308.7
C	2,000	(385)	(385)	(385)
	1,750	422.2	427.8	425.4
	1,500	452.6	463.2	457.3
	1,250	477.1	492.1	481.7
	1,000	496.3	514.9	499.6
	500	520.5	544.2	514.2
	0	528.1	553.6	523.3

amined in the present work also. Results for the cases of $m=2$ and $m=3$ are given in Table 1 together with the Haefeli's best case of $n=4$. As shown in Fig. 4, there is practically no difference between the present $m=3$ model and the Haefeli's $n=4$ model for $x=0\sim400$ km or $h=1{,}800\sim3{,}160$ m. It may be concluded therefore that the present $m=3$ model well fits the observed profile of Greenland ice sheet. Dealing in more detail with the curve fitting between the present $m=3$ model and the observed profile, it may be found in Fig. 4 that the $m=3$ model can well represent the observed profile down to $h\simeq1{,}500$ m, but h of the theoretical model becomes higher than that of observed data for h below 1,500 m. It seems therefore that the present theoretical result is more feasible than the Haefeli's $n=4$ model which results in theoretical h values smaller than the observed values in the outer region.

7. Concluding Remarks

As already discussed in the preceding sections, the significant difficulty in the Nye's and Haefeli's models is mainly that the ice mass conservation does not hold within their ice sheet models and consequently they cannot have their steady state. This is due to the theoretical result in their models that the horizontal velocity of ice flow (u in the Nye's model and $u_m=\frac{1}{h}\int_0^h u dz$ in the Haefeli's model) becomes infinitely large with x approaching to L. In these models u is given by

$$u\,(\text{or}\ u_m)=\frac{bx}{h}=b\left(\frac{L}{H}\right)\frac{\xi_o}{\eta_o}, \tag{38}$$

whereas u in the present steady state model is expressed as

$$u=\left[\frac{m+1}{m}\,b\,\beta\,(H-h)\right]^{\frac{m}{m+1}}=\left(\frac{2m+1}{m+1}\right)b\left(\frac{L}{H}\right)(1-\eta_o)^{\frac{m}{m+1}}. \tag{39}$$

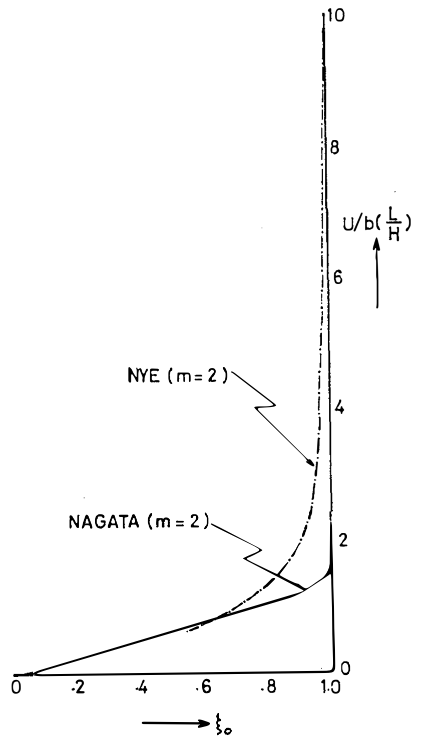

Fig. 5. Horizontal velocity of ice flow dependent of $x/L=\xi$.

In Fig. 5, values of u as a function of ξ_o are illustrated for the case of $m=2$ in both the present and Nye's models. The relation of u_m to ξ_o in the Haefeli's model is almost the same as that in the Nye's model.

The u values in the Nye's and Haefeli's models are not much different from those in the present steady state model for $\xi_o < 0.4$, but the former becomes considerably larger than the latter for the range of $\xi_0 > 0.5$, finally approaching to infinity with ξ_0 approaching to 1. The surface integral to represent the total net accumulation rate of ice over the whole ice sheet surface, $I_N\,(L_N)$, expressed by eq. (16′), can be analytically obtained, the results of calculation showing $I_N(L_N)=-\infty$ regardless of m. The divergence of integration is due to that of the second term of eq. (16′), which represents the outflow of ice through the ice sheet surface.

As far as the two-dimensional ice sheet model of steady state is concerned, the present model only may be considered self-consistent with respect to all physical boundary conditions. However, an extension of the present self-consistent theory to a three-dimensional circular ice sheet or a modification of the Haefeli's model on the basis of ice mass conservation law cannot keep the mathematical simplicity such as used in the present work.

References

HAEFELI, R. (1961): Contribution to the movement and the form of ice sheets in the Arctic and Antarctic. J. Glaciol., **3**, 1133–1150.

NAGATA, T., FISHER, R.M. and SUGIURA, N. (1975): Metallographic and magnetic properties of a Yamato iron meteorite —Yamato-75-105. Mem. Natl Inst. Polar Res., Ser. C, **10**, 1–11.

NYE, J.F. (1959): The motion of ice sheets and glaciers. J. Glaciol., **3**, 493–507.

PATERSON, W.C.B. (1969): The Physics of Glaciers. Oxford, Pergamon Press, 241p.

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