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A FIRST-STEP IMPROVEMENT OF SPLINE FUNCTION METHOD IN IONOSPHERIC TOMOGRAPHY (EXTENDED ABSTRACT)

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It is almost a decade ago that the ionospheric application of computerized tomography using radio transmissions from beacon satellites was proposed (AUSTEN *et al.*, 1986). The method involves inversion of total electron content (TEC) measurements to reconstruct a two-dimensional image of the ionospheric electron density.

There are several algorithms which are applicable to the present issue (*e.g.*, OKUZAWA *et al.*, 1995; RAYMUND, 1995). Once we adopt the *spline function method* (SFM) among them, we encounter a serious problem that we need additional data of TEC along horizontal direction, which will be basically due to the nature of restricted imaging system. In order to overcome this defect, we attempt in the present study to introduce a kind of simple filtering which suppresses the amplification of equivalent noise involved in the solution of, what is called, *normal equation* that is derived on the way to minimizing the relevant noise term in least squares sense.

Let the region of the ionosphere for imaging be partitioned into K_1 (vertical) $\times$ K_2 (horizontal) pixels, each side of the region being parallel to the θ=*const.* lines and r=*const.* curves in a two-dimensional polar coordinate system with the origin at the center of the earth. Within each pixel the electron density is assumed to be a constant value. Similarly, let several receiving stations be located on the ground, and a certain number of TEC data be obtained at the p-th receiving stations.

Expressing the q-th TEC data obtained at the p-th station as TEC_{pq}, we can write as follows:

$$TEC_{pq}=\sum_{k=1}^{K_1}\sum_{l=1}^{K_2} L_{pq,\,kl}\,N_{kl}+\varepsilon_{pq}\,, \tag{1}$$

where $L_{pq,\,kl}$ stands for the length of the (p, q)-th line-of-sight passing the (k, l)-th pixel as a cross section, and the term of ε_{pq} is the residual originating from noises, measurement and/or digitizing errors. Our objective is to estimate N_{kl} from a given TEC_{pq} (*i.e.*, observed quantity) with respect to a known set of $L_{pq,\,kl}$.

We assume that the electron density $N(r, \theta)$ is a two-dimensional spline function expressed as a linear combination of the products of one-dimensional normalized B-spline, $N_{mi}(r)$ and $N_{mj}(\theta)$ of order m (degree $m-1$), as follows:

$$N(r, \theta)=\sum_{i=1}^{n+m}\sum_{j=1}^{n'+m} c_{ij}N_{mi}(r)N_{mj}(\theta)\,, \tag{2}$$

where n and n' denote the number of so-called knots or joints located in the form of

lattice in the direction of r and θ, respectively, and c_{ij} stands for the spline coefficients to be determined.

Substituting eqs. (2) into (1), and then in order to minimize the square of ε_{pq} term in eq. (1), we set the differentiation of it zero. As a result, we obtain a well known *normal equation* as follows:

$$\sum_{i=1}^{n+m}\sum_{j=1}^{n'+m}\left[\sum_{p=1}^{M_1}\sum_{q=1}^{M_2(p)}\varphi_{ij}(p,q)\varphi_{rs}(p,q)\right]c_{ij}=\sum_{p=1}^{M_1}\sum_{q=1}^{M_2(p)}TEC_{pq}\varphi_{rs}(p,q)\,,$$

$$\varphi_{ij}(p,q)=\sum_{k=1}^{K_1}\sum_{l=1}^{K_2}L_{pq,\,kl}N_{m,\,i}(r_k)N_{m,\,j}(\theta_l)\,,$$

$$(r=1,\,2,\,\cdots,\,n+m;\ s=1,\,2,\,\cdots,\,n'+m) \tag{3}$$

where M_1 and M_2 (p) denote the numbers of radio stations on the ground and radio ray paths pertinent to p-th station,respectively.

The point of the present issue is to solve eq. (3) with respect to c_{ij}. For convenience, the eq. (3) can often be written in a simple matrix form as

$$\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}\,. \tag{4}$$

It is not so easy to solve this equation with respect to $\boldsymbol{x}$, *i.e.*, c_{ij}, because it is still *ill-posed* unless some additional *TEC* data are taken into account. We will, therefore, use in this study a simple regularization method briefly described below.

In eq. (4), the coefficients matrix $\boldsymbol{A}$ is decomposed into the products of a diagonal singular-value matrix $\boldsymbol{\Lambda}$ and eigenvectors in matrix form, $\boldsymbol{U}$ and $\boldsymbol{V}$, such that:

$$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^T=\sum_{i=1}^{M}\boldsymbol{u}_i\lambda_i\boldsymbol{v}_i^T\,, \tag{5}$$

where, $M=(n+m)\times(n'+m)$, singular vectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are the columns of the matrices $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively, λ_i are the singular values of $\boldsymbol{A}$, and the superscript T means "transposed". The singular vectors are orthogonal, $\boldsymbol{u}_i^T\boldsymbol{u}_j=\boldsymbol{v}_i^T\boldsymbol{v}_j=\delta_{ij}$, for $i, j=1,\,2,\,\cdots,\,M$, and the singular values of λ_i are nonnegative and appear in nonincreasing order: $\lambda_1\geq\lambda_2\geq\lambda_3\geq\cdots\geq\lambda_M\geq 0$. In practice, some values of λ_i (in the present case, for $i\geq 33$) can be very small, and so $\lambda_i^{-1}\gg 1$, then the solution of eq. (4), $\boldsymbol{x}=(\sum_{i=1}^{M}\boldsymbol{v}_i\lambda_i^{-1}\boldsymbol{u}_i^T)\boldsymbol{b}$, may have a diverging solution.

One way of suppressing the undesirable effect of the small singular values is to modify the term $\boldsymbol{\Lambda}$ to $\boldsymbol{\Lambda}+\alpha\boldsymbol{I}$, where $\boldsymbol{I}$ denotes the unit diagonal matrix, and α stands for a compensative numerical factor; introduction of this factor can suppress the amplification of equivalent noise appearing in the solution of the *normal equation*. The approximate solution of $\boldsymbol{x}$ following this procedure is called 'damped least squares' by LEVENBERG (1944).

Although we have not yet found any optimal value of α from the original scheme of LEVENBERG (*i.e.*, we are still groping for the way how to evaluate the equivalent-noise variance in the *normal equation*), we have succeeded in getting a non-diverging solution of eq. (3) by applying the above method without assuming any additional *TEC* data: A typical example of the reconstructed images on the same simulation condition of OKUZAWA *et al.* (1995), is illustrated in the following.

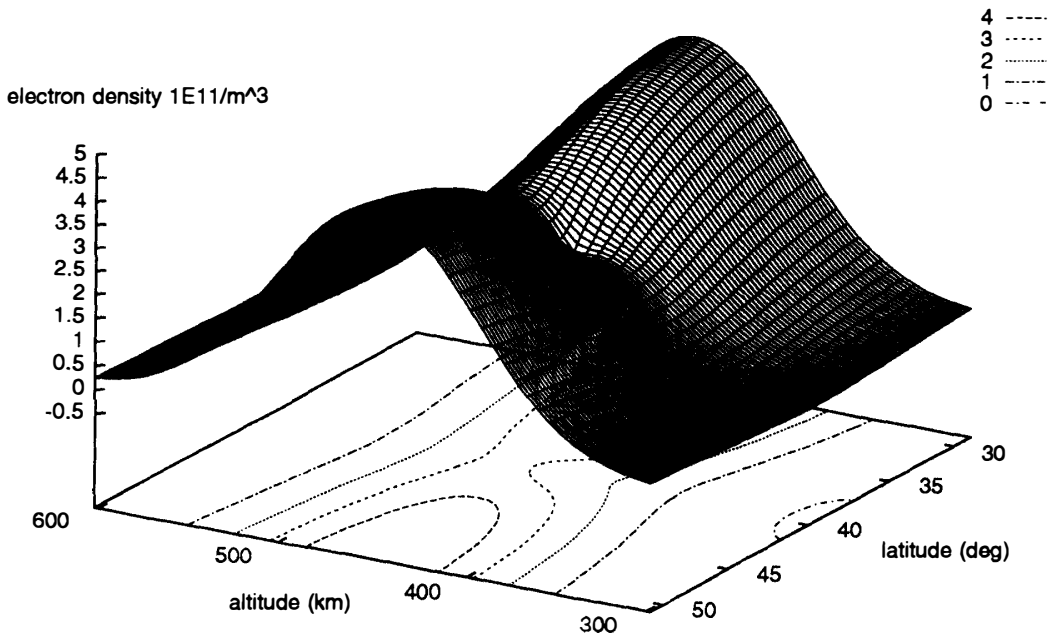

Fig. 1. Electron density profile used as an original model.

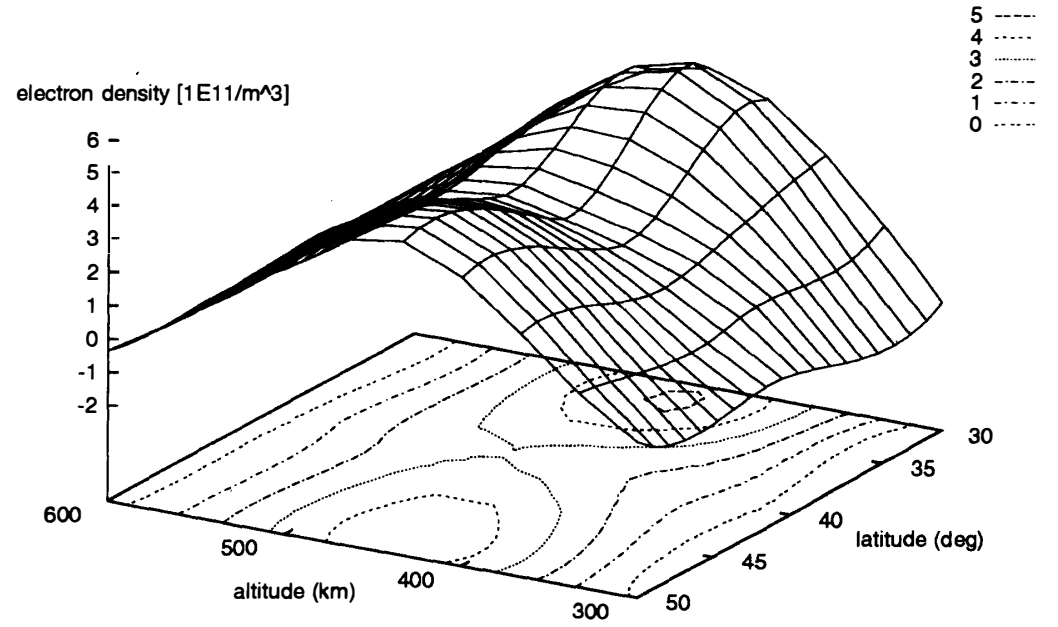

Fig. 2. An example of the image profile reconstructed with SFM.

Figure 1 shows the electron density profile used as an original model, and Fig. 2 is the image profile reconstructed by assuming $m=4$ and $n=n'=2$ and $\alpha=2.5$ for the present calculation. The rate of reconstruction errors, $\sqrt{\frac{\sum_{k,l}(N_{kl}-N_{kl}^{cal})^2}{\sum_{k,l}N_{kl}^2}}$, amounts to about 15% in a major part of the projection region. This value is to be reduced if we can solve such crucial problems as those

described below.

Finally, we note the problems left for our future research as follows; (i) confirmation of the effectiveness of any information criterion that prescribes the optimum number of knots and the order of B-spline, (ii) establishment of the way by which we can optimize the constant factor α that has been assumed arbitrarily in the present paper, and (iii) comparison of the solutions between ours and other methods, *e.g.*, TSVD (truncated singular value decomposition) or its modification (HANSEN *et al.*, 1992).

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