DISCRETE ORDINATE SOLUTIONS OF THE TRANSPORT EQUATION FOR AURORAL ELECTRONS

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Abstract: Presented here is an interpretation of the discrete ordinate method which has been used to solve the transport equation of auroral electrons in the atmosphere, and the accuracy of this method is discussed with different orders of approximations by numerical analysis. The discrete ordinate method is based on an implicit assumption on the angular distribution of electron intensity. The three types of two-stream approximations (single-Gauss, double-Gauss and Eddington's approximations) are characterized by the different assumptions on which each type is based. The error analysis is given on the energy deposition rate and the energy albedo. The double-Gauss 4-stream approximation is found to be practical in obtaining these two quantities with the accuracy within a few percent, whereas the error in the single-Gauss 4-stream approximation amounts to about 20% for the energy albedo. For the energy deposition rate, any of the two-stream approximations for the incident energies of keV regions has the accuracy comparable to that of the calculation without taking electron scattering into account. For the energy albedo, the errors in the two-stream approximations exceed about 30% in the same energy range.

1. Introduction

Among the various attempts to describe the collisional interaction of auroral electrons with the neutral atmosphere, a method to solve the full transfer equation has been developed for more than a decade. In this method, discrete energy losses of electrons due to inelastic collisions and wide-angle scattering due to elastic collisions are taken into account for a wide energy range in a single manner. BANKS and NAGY (1970), and NAGY and BANKS (1970) formulated an equation in a two-stream approximation for electrons with energies below 100 eV, where scattering through on elastic collision is fairly isotropic. MANTAS (1975), and MANTAS and WALKER (1976) carried out the same kind of calculation in a multi-stream approximation for electrons in the same energy range. Using the multi-stream approximation (20 grid points in μ-space, where μ is the cosine of the electron pitch angle), STRICKLAND *et al.* (1976) extended the energy range up to a few tens of keV, where elastic scattering of electrons is strongly anisotropic. In these multi-stream approximations, intervals between neighboring μ-grid points seem to be constant. On the other hand, STOLARSKI (1972) applied the discrete ordinate method, in which the Gaussian quadrature formula plays an important role, to the problem of photoelectron transport in the atmosphere. This method was used widely in the field of radiative transfer. STOLARSKI (1972), however, did

not mention its superiority resulting from the use of the Gaussian quadrature formula. The discrete ordinate method was further discussed by STAMNES (1977) for the transport problem of photoelectrons and was applied to auroral electrons by STAMNES (1980). STAMNES (1980, 1981) discussed the validity of the two-stream approximation which is the lowest order approximation of the discrete ordinate method, and showed that the two-stream approximation would provide adequate results, especially in the estimation of backscattered and height integrated quantities. However, his work has left a few points to be discussed further. First, as regards height profiles of the electron intensity, evaluation of accuracy in several orders of approximations is made for "non-degraded" (denoted as "non-absorbed" by STAMNES, 1980) electrons, which keep their initial energies in successive collisional processes, in the collision depth space. The argument about convergency of the solution, which includes the effect of degraded electrons, with various orders of approximations was given by TAKAHASHI (1975), but it was made only for the primary energy of 200 eV. Second, although there are three types of two-stream approximations, as will be shown in the next section, the relationship among these types discussed by STAMNES (1977, 1980, 1981) seems to include some confusion in making proper choice of a set of parameters.

It is necessary to include the effect of multiple scattering of electrons for modeling of interaction between auroral electrons and the atmosphere. However, since high order multi-stream calculations require many computational tasks, it is desirable to use the low order approximation as far as possible. The purpose of this paper is to clarify the assumptions on which the discrete ordinate method is based and, in particular, to describe the distinction among different types of two-stream approximations. Furthermore, the accuracy of the discrete ordinate method with different orders of approximations is discussed in detail.

2. Discrete Ordinate Solutions

The equation for the transport of electrons along a magnetic line of force in the atmosphere is given by (STAMNES, 1980)

$$\mu\frac{\mathrm{d}}{\mathrm{d}\tau}I(\tau, E, \mu)=I(\tau, E, \mu)-\frac{\omega_e(\tau, E)}{2}\int_{-1}^{1}p_e(\mu, \mu')I(\tau, E, \mu')\mathrm{d}\mu' - \int_{E'>E}\mathrm{d}E'R(\tau, E, E')\frac{1}{2}\int_{-1}^{1}p_{ie}(\mu, \mu')I(\tau, E', \mu')\mathrm{d}\mu' , \quad (1)$$

where

$I(\tau, E, \mu)$: intensity of electrons in unit of $\mathrm{cm}^{-2}\,\mathrm{sr}^{-1}\,\mathrm{s}^{-1}\,\mathrm{eV}^{-1}$,
E: electron energy,
τ: collision depth,
ω_e: single scattering albedo,
$p_e(\mu, \mu')$: phase function, *i.e.*, normalized differential cross section for elastic scattering,
$p_{ie}(\mu, \mu')$: phase function for inelastic scattering,
$R(\tau, E, E')$: normalized redistribution cross section in energy space for inelastic collision at τ.

The second term on the right-hand side denotes multiple scattering due to elastic collisions and the third term describes cascading from higher energies.

For simplification, we assume a single-constituent atmosphere and consider only "non-degraded" electrons in this section. Then eq. (1) becomes

$$\mu \frac{\mathrm{d}}{\mathrm{d}\tau} I(\tau, \mu) = I(\tau, \mu) - \frac{\omega_e}{2} \int_{-1}^{1} p_e(\mu, \mu') I(\tau, \mu') \, \mathrm{d}\mu' \,. \tag{2}$$

In this equation, the phase function can be expanded into a series of Legendre polynominals in the form

$$p_e(\mu, \mu') = \sum_{l=0}^{\infty} (2l+1) \chi_l P_l(\mu) P_l(\mu') \,, \tag{3}$$

where

$$\chi_l = \int_{4\pi} p_e(\cos \Theta) P_l(\cos \Theta) \frac{\mathrm{d}\Omega}{4\pi} \,,$$

and Θ is the scattering angle. The procedure of the discrete ordinate method is given by STAMNES (1980) and STAMNES and SWANSON (1981) in detail.

The discrete ordinate method of order n, that is $2n$-stream approximation, is well known as a procedure by which an integro-differential equation is reduced to a simultaneous system of $2n$ ordinary differential equations. In this method, the integral term in eq. (2) is replaced by a sum of terms multiplied by weighting factors inherent in the quadrature rule. When the Gaussian formula is applied to the whole range in μ-space, replacement of an integral term by a summation is an exact transformation if the integrand, $p_e(\mu, \mu') I(\mu')$, is a polynomial of order less than or equal to $4n-1$ in μ'. This means that the electron intensity is assumed to be a function of μ of order $2n$ when the known quantity $p_e(\mu, \mu')$ is approximated by a function of order $2n-1$. In order to make this approximation on $p_e(\mu, \mu')$, we truncate the order of expansion in eq. (3) by $2n-1$.

For $n=1$, that is the two-stream approximation, eq. (2) reduces to

$$\bar{\mu} \frac{\mathrm{d}I^+(\tau)}{\mathrm{d}\tau} = [1 - \omega_e(1-\beta)] I^+(\tau) - \omega_e \beta I^-(\tau) \,, \tag{4}$$

$$-\bar{\mu} \frac{\mathrm{d}I^-(\tau)}{\mathrm{d}\tau} = [1 - \omega_e(1-\beta)] I^-(\tau) - \omega_e \beta I^+(\tau) \,, \tag{5}$$

where $\bar{\mu} = 1/\sqrt{3}$, $I^{\pm}(\tau) = I(\tau, \pm\bar{\mu})$ and

$$\beta = \frac{1}{2}(1 - \chi_1) \,. \tag{6}$$

According to the above interpretation, $I(\mu)$ is assumed to be a linear function of μ and is not regarded to be hemispherically isotropic. Here, $\pm\bar{\mu}$ are merely the quadrature points and not the mean values of μ. Similarly, $I^{\pm}$ are the values of $I(\mu)$ at $\mu = \pm\bar{\mu}$ and not the upward- and downward-hemispherically mean values of the intensity.

On the other hand, in the two-stream approximations, there is an approximation called the Eddington's two-stream approximation which is independent of the discrete ordinate method and is based on the assumption that the electron intensity is hemispherically isotropic (*e.g.* GOODY, 1964). Without any other assumptions, in other words, without the approximation on the phase function, we can obtain a set of equations formally identical to eqs. (4) and (5) from eq. (2). In this approximation, however, $\bar{\mu}$ is the mean value of μ being equal to 1/2 and

$$\beta = \frac{1}{2}\int_0^1 d\mu \int_{-1}^0 d\mu' p_e(\mu, \mu') \ .$$

$I^{\pm}$ are the upward- and downward-electron intensity and β denotes the probability that an electron is scattered from one of the hemispheres to the other through an elastic collision with a neutral particle. It should be noted that, strictly speaking, β given by eq. (6) does not have the same meaning as that shown immediately above.

In the above discussion on the discrete ordinate method, the Gaussian formula is applied to the entire range of μ ($-1 \leq \mu \leq 1$) (hereafter referred to as single-Gauss). However, it is also possible to apply the Gaussian formula separately to the half ranges, $-1 \leq \mu < 0$ and $0 < \mu \leq 1$ (SYKES, 1951; STAMNES, 1977) (hereafter referred to as double-Gauss as denoted by SYKES, 1951). In the same manner as the single-Gauss approximations, it will be shown that, for the double-Gauss $2n$-stream approximation, the electron intensity is assumed to be functions of μ of degree $n-1$ in the plus and minus μ-regions separately under the approximation that the phase function is a polynomial of μ of degree n. For isotropic scattering, SYKES (1951) showed that the double-Gauss approximation is superior to the same order single-Gauss approximation and STAMNES (1977) also obtained the similar result. However, this conclusion is not necessarily valid for anisotropic scattering, partly because the phase function must be truncated for this case, whereas the approximation of phase function is not required for isotropic scattering. The reason why the double-Gauss method was introduced is the fact that, at $\mu=0$, the electron intensity has a discontinuity at $\tau=0$ and is not smooth at shallow τ.

In the double-Gauss two-stream approximation, similarly to the single-Gauss case, we can also obtain a set of equations formally identical to eqs. (4) and (5) from eq. (2), where $\bar{\mu}$ denotes the quadrature point and has a value of 1/2, and $\beta=(1-3\chi_1/4)/2$. In this approximation, the phase function is expressed as a linear function of μ. However, noting that the electron intensity is regarded to be isotropic in each of the hemispheres, the truncation of the phase function is no more required. Consequently, the double-Gauss two-stream approximation is included in the Eddington's two-stream approximation in principle.

3. Results

For numerical analysis, this paper assumes simply a nitrogen molecule atmosphere in which the distribution of the number density is identical to that of the total density given in the CIRA (1972) mean atmosphere. All results shown below are obtained for the incidence of mono-energetic electrons at the top of the atmosphere.

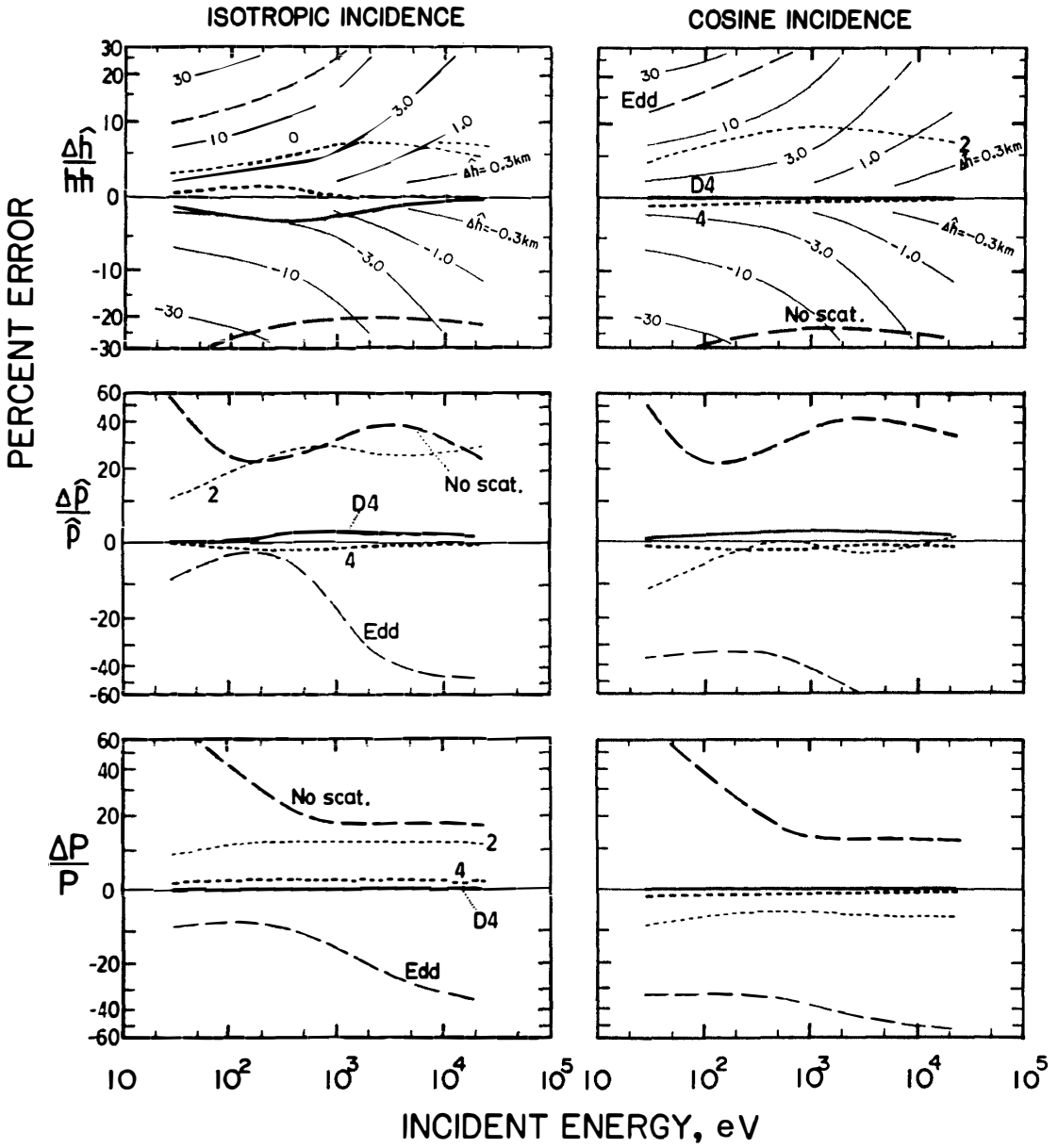

Fig. 1. Relative errors in the following quantities as functions of the incident electron energy: the height, $\hat{h}$, at which the energy deposition rate in the neutral atmosphere reaches its maximum (uppermost), the peak value of the energy deposition rate, $\hat{p}$ (middle), and the height-integrated energy deposition rate, P (bottom). The peak height error, $\Delta\hat{h}$, is normalized by the half maximum width, hh, of the energy deposition profile. The numbers 2 and 4 refer to the single-Gauss 2- and 4-stream approximations, respectively. D4 denotes the double-Gauss 4-stream approximation and Edd is the Eddington's 2-stream approximation. The results of the calculation in which the scattering effect is not taken into account are also shown (No scat.). The isotropic electron incidence and the cosine dependent incidence are assumed for the left- and right-hand side figures, respectively.

The effects of secondary electrons and degraded primary electrons are included unless otherwise noted.

In Fig. 1, the relative errors estimated in quantities concerning the energy deposi-

tion rate in the neutral atmosphere are shown as functions of the incident electron energy for several orders of approximations for two boundary conditions; *i.e.*, an isotropic electron incidence and a cosine dependent incidence. The uppermost panels show the errors in the height, $\hat{h}$, at which the energy deposition rate reaches its maximum. These errors are normalized by the half maximum widths of the energy deposition profiles derived from the double-Gauss 8-stream approximation. Absolute values of height deviations, $\Delta\hat{h}$, are also shown by contour lines. The errors in the peak value of the energy deposition rate, $\hat{p}$, are given in the middle panels, and the bottom panels show those in the height-integrated energy deposition rate, P. For reference, the results obtained by neglecting scattering effects are also shown by the heavy dashed lines. In evaluating the errors, the results from the double-Gauss 8-stream approximation are taken as the standards, because in the preliminary calculation for "non-degraded" electrons the relative errors in the double-Gauss 8-stream approximation were found to be within about 0.1% for the quantities concerning the energy deposition rate and within about 1% for the backscattered flux as compared with the double-Gauss 32-stream approximation. Concerning the height-integrated energy deposition rate, the double-Gauss 4-stream approximation gives the results with an accuracy within 0.5% in the full energy range, whereas those of the single-Gauss 4-stream approximation have an accuracy within about 2%. On the other hand, the errors in results from the single-Gauss two-stream approximation are about 10% and those from the Eddington's approximation amount to several tens of

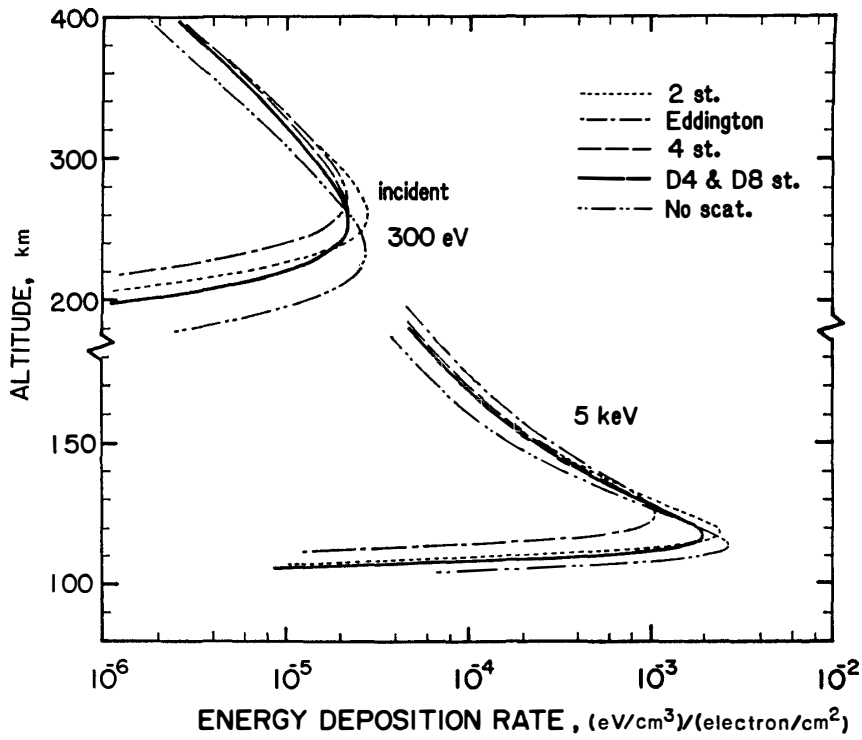

Fig. 2. Energy deposition profiles per unit number flux of incident electrons. The results calculated from several approximations are shown for the isotropic incidence of mono-energetic electrons at 300 eV and 5 keV.

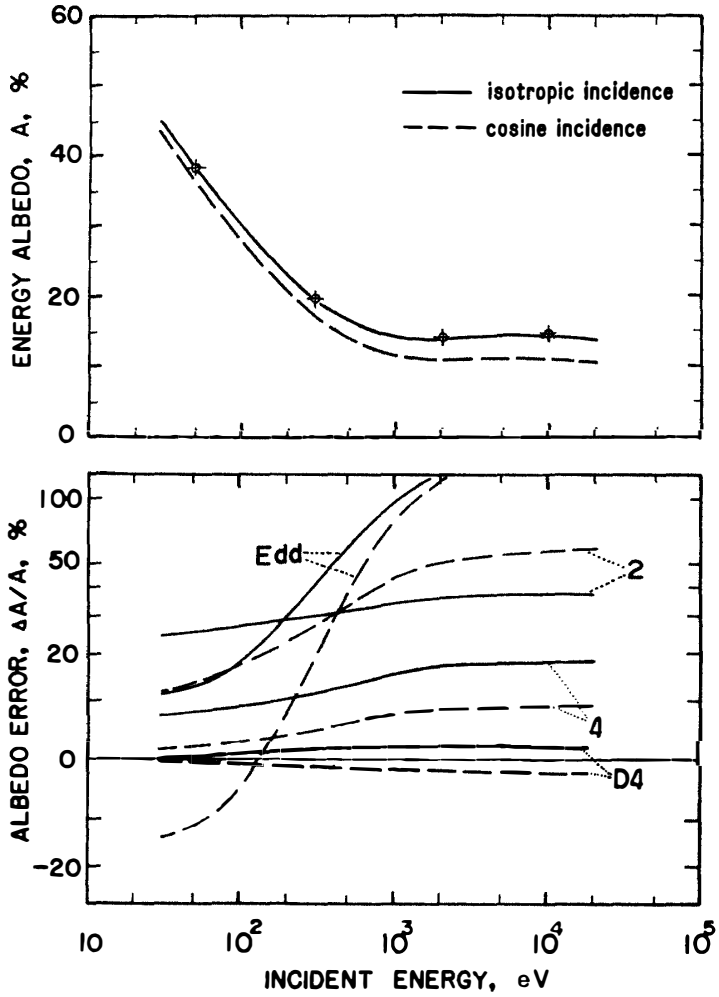

Fig. 3. Upper panel: Energy albedos, which include the contribution of secondary electrons and degraded electrons, shown as functions of the incident electron energy. Solid line (isotropic incidence) and dashed line (cosine incidence) are calculated from the double-Gauss 8-stream approximation. Symbol ⊕ shows the results from the double-Gauss 4-stream approximation for the isotropic incidence.
Lower panel: Relative errors in the albedo for several approximations for the isotropic (solid lines) and the cosine dependent (dashed lines) incidence.

percent at higher incident energies. The errors in the two-stream approximations for keV electrons are quite comparable to those in the calculation which assumes no-scattering. For the two quantities, $\hat{h}$ and $\hat{p}$, shown in the upper and the middle panels, the similar conclusion to the above is obtained except that the single-Gauss and the double-Gauss 4-stream approximations have the same degree of accuracy with the maximum error of about 3 %.

The profiles of the energy deposition rate for unit flux (1 electron/cm^2 s) incidence of isotropic and mono-energetic electrons (300 eV and 5 keV) are shown in Fig. 2. The result of the double-Gauss 4-stream approximation agrees well with that of the double-Gauss 8-stream approximation which is, as mentioned earlier, sufficiently accurate.

In the upper panel of Fig. 3, the energy albedos are shown as functions of the in-

cident energy for the double-Gauss 8-stream approximation, where the energy albedo is defined as the ratio of the energy flux of the backscattered electrons to that of incident electrons. The albedo errors in several orders of approximations are given in the lower panel. The errors resulting from the two-stream approximations are fairly large and, in particular, the Eddington's two-stream approximation is quite inaccurate at higher incident energies. The double-Gauss 4-stream approximation is always accurate within about 2%, whereas the errors in the single-Gauss 4-stream approximation become large towards higher incident energies, being about 20% at 10 keV. Even if the maximum error in the double-Gauss 8-stream approximation which was adopted to obtain the standards is as large as 1%, the maximum error in the double-Gauss 4-stream approximation would not exceed about 3%.

4. Discussion

In Section 2, it was mentioned that the double-Gauss two-stream approximation is included in the Eddington's two-stream approximation. This means that the double-Gauss two-stream treatment is obtained by adding an approximation of the phase function to the Eddington's two-stream approximation. Therefore, in principle, the Eddington's approximation is thought to be more accurate than the double-Gauss two-stream approximation. However, as shown in Table 1 which gives the results from the three types of two-stream approximations for "non-degraded" electrons, we obtained the results contrary to the above expectation. This may be partly due to the situation that, in the double-Gauss two-stream approximation, an error due to the approximation to the phase function cancels an error arising from the assumption that the electron intensity is isotropic in each of the hemispheres. It is noted, however, that the double-Gauss two-stream treatment has, in general, only the same degree of accuracy as the single-Gauss two-stream approximation.

Finally, we will briefly discuss the δ-M method which was developed by WISCOMBE (1977) in order to obtain a better approximation to the phase function than the ordinary

Table 1. Comparison among three types of the two-stream approximations for "non-degraded" electrons for the isotropic incidence. The "non-degraded" electrons refer to the electrons within the specified energy cell. The energy deposition rate here concerns only "non-degraded" electrons. The results from the double-Gauss 32-stream approximation are also shown as the standards.

	Energy deposition peak height (km)		Maximum deposition rate (arbitrary unit)		Flux albedo	
Incident energy	$E=10$ keV	500 eV	10 keV	500 eV	10 keV	500 eV
Width of energy cell	($\Delta E=1$ keV)	(50 eV)	(1 keV)	(50 eV)	(1 keV)	(50 eV)
SG 2-stream	117.8	297	1.20	1.25	5.55 (−2)	4.46 (−2)*
DG 2-stream	119.4	306	0.95	1.06	6.32 (−2)	4.73 (−2)
Eddington	122.1	309	0.75	1.03	17.8 (−2)	8.13 (−2)
DG 32-stream	116.2	293	1.00	1.00	3.77 (−2)	3.91 (−2)

* A(−2) reads $A\times10^{-2}$.

technique by introducing the Dirac delta function in the Legendre polynominal expansion. In the calculations shown above, the δ-M method is used only for the double-Gauss approximations, because the preliminary calculations showed the result that although the δ-M method works in the double-Gauss approximations, it is not effective in principle in the single-Gauss approximations. It was also shown by the same calculation that the superiority of the double-Gauss 4-stream approximation to the single-Gauss 4-stream approximation in some quantities, as discussed in the previous section, is attributable to the use of δ-M method.

In conclusion, an explanation of the discrete ordinate method has been presented and the accuracy of this method has been discussed for electrons with incident energies below 20 keV. Each type of two versions of the discrete ordinate method (single-Gauss and double-Gauss) is based on its own assumptions with regard to the phase function and the angular distribution of electron intensity. The double-Gauss 4-stream approximation is shown to be a good approximation with the precision within a few percent and is a little more accurate than the single-Gauss 4-stream approximation. On the other hand, any type of the two-stream approximations has the errors amounting to or exceeding 20–30% in some cases.

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