A SIMPLE METHOD FOR CALCULATING MASS FLUX IN AN ICE SHEET, WITH A CONSIDERATION OF ITS TEMPERATURE PROFILE

Masatoshi NAGAO, Masayoshi NAKAWO and Akira HIGASHI

Department of Applied Physics, Faculty of Engineering, Hokkaido University, Kita-13, Nishi-8, Kita-ku, Sapporo 060

Abstract: For estimating the mass flux of ice at a given place on an ice sheet, a new method has been proposed, in which the vertical temperature profile was taken into account. The linear profile was assumed and it was determined from the bottom temperature and the temperature gradient at the bottom, both of which were estimated by the column model. The temperature dependence of the flow law of ice was converted to a simple expression which approximates the usual Arrhenius type relations well. This conversion together with the above assumption enabled the calculation of the mass flux to be done without any numerical integration. Error in the estimate of the mass flux caused by the linear temperature profile was calculated for conditions in the Shirase drainage basin, East Antarctica and it was shown to be less than about 11%.

1. Introduction

For computer simulation of the dynamical behavior of large ice sheets, the Mahaffy's model (MAHAFFY, 1976) is one of the most advanced, because it is a three-dimensional model which can be used for a non-steady state, taking into account the bedrock topography. Adopting this model, NAGAO and NAKAWO (1982) tried to simulate the variations of the profile of the ice sheet to be caused by variations of the net balance, the surface temperature and the bedrock topography. In this modeling, the temperature variation in the ice sheet was ignored as in the previous cases. The model assumed unrealistically that the ice temperature was constant vertically. However, the computation should take into account the temperature profile, because the stress-induced flow of polar ice sheets is highly dependent on temperature.

In this paper, a simple method is proposed for obtaining the mass flux of ice, an essential quantity for ice sheet dynamics. For the depth profile of temperature at a given place on an ice sheet, a linear profile is assumed which is formulated with the bottom temperature and the temperature gradient at the bottom. The bottom temperature can be derived by solving the equation of heat conduction including the horizontal advection term (RADOK *et al.*, 1970). The temperature dependence of the flow law of ice was converted from the Arrhenius equation to a simple expression. This conversion together with the above assumption of temperature profile in an ice sheet makes the calculation of the mass flux very simple. Actually, we do not need numerical integration since it can be carried out analytically. This simplification can save a large amount of computation in simulating the dynamical behavior of an ice sheet in a three-dimensional model.

2. Equation of the Mass Flux in the Ice Sheet

The coordinate system used in calculation of the mass flux is shown in Fig. 1. The x and y axes are taken perpendicular to each other in the horizontal plane,

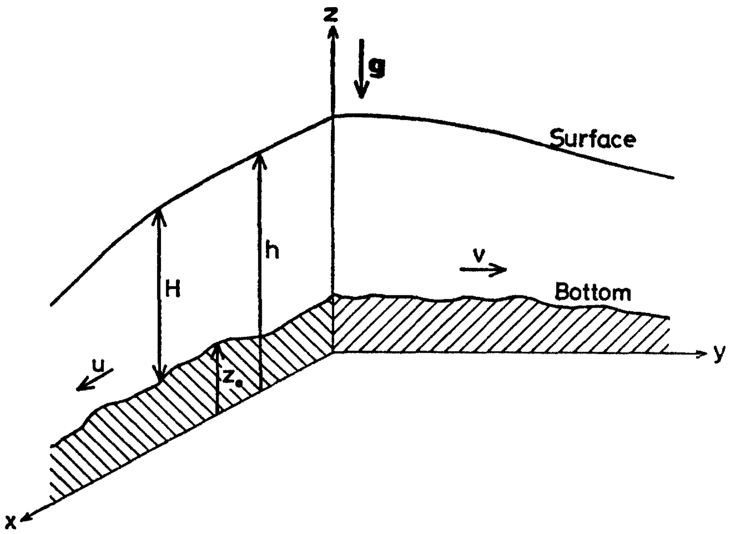

Fig. 1. Coordinate system used. z axis is parallel to the gravity vector. h and z_0 denote respectively the height of the ice surface and the bedrock above the standard horizontal plane (xy). Ice thickness is given by H ($H=h-z_0$). u represents the velocity of the ice flow in the x direction and v that in the y direction.

and the z axis vertical and positive upward. Notations h, z_0 and H shown in this figure are for the height of ice surface, bedrock and ice thickness respectively. The mass flux in the x direction q_x obtained by MAHAFFY (1976) was

$$q_x = -2\bar{A}(n+2)^{-1}(\rho g)^n \alpha^{n-1}\left(\frac{\partial h}{\partial x}\right)H^{n+2} + u_b H. \tag{1}$$

Here $\bar{A}$ is the flow law parameter averaged over the depth and assumed to be the same for all x and y (PATERSON, 1981). In this equation and in the next improved equation, n is the exponent in a flow law of ice of the type $\dot{\varepsilon} = A\tau^n$ and ρ the density of ice. α is the slope of the surface, given by

$$\alpha = \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}.$$

Since the flow law parameter $\bar{A}$ depends upon temperature, contributions of this term should be included in integrations both for obtaining $u(z)$ from the flow law and computing q_x from u. We developed the mathematics as follows:

$$q_x = -2(\rho g)^n \alpha^{n-1} \frac{\partial h}{\partial x} H^{n+2} I + u_b H \,, \tag{2}$$

where

$$I = \int_0^1 \int_0^l A(1-l')^n \mathrm{d}l' \mathrm{d}l, \tag{3}$$

in which A is the temperature dependent parameter in the flow law $\dot{\varepsilon} = A\tau^n$. Instead of the real height z, the normalized height $l = (z - z_0)/H$ was introduced. With this height, the integrations from z_0 to z for $u(z)$ and z_0 to $z_0 + H$ for q_x were converted from 0 to l and from 0 to 1 respectively. In the derivation of eq.(2), ρ and n were assumed constant all over in the ice sheet. For calculating q_x, taking account of the temperature profile, one has to know the temperature dependence of A in eq.(3). Then, if it is converted to a simple function of the depth using a suitable approximation of the depth profile of temperature in an ice sheet, the function I could be obtained analytically without numerical integrations. Once the value of I is estimated, q_x would be calculated with eq.(2), although u_b has to be given independently unless it is zero.

3. Temperature Dependence of the Flow Law Parameter A

Many experimental data suggest that over the stress range important in glacier flow, the exponent n is approximately 3 and does not depend upon temperature (PATERSON, 1981). On the other hand, the parameter A is usually expressed by the Arrhenius equation,

$$A = A_1 \exp\left(-\frac{Q}{RT}\right), \tag{4}$$

where, A_1 is a constant, R is the gas constant, and T is the absolute temperature. The value of Q, the creep activation energy, is considered to be about 60 kJ mol^{-1} below -10°C and 139 kJ mol^{-1} between -10 and 0°C (PATERSON, 1981), *i.e.*, Q is also temperature dependent. This fact makes it inconvenient to use eq.(4) for the calculation of I expressed by eq.(3). It is considered more practical, hence, to introduce another expression which can approximate the above dependence in a temperature range between -30°C and 0°C, which is the one usually encountered in real ice sheets in Antarctica.

An expression

$$A = \frac{A_0}{(T_0 - T)^3}, \tag{5}$$

is proposed. Two arbitrary constants A_0 and T_0 are determined so that the curve (solid line in Fig. 2) will be a good approximation to the one given by the Arrhenius relations (dotted line in Fig. 2). A_0 is 4.2×10^{-12} K^{3}kPa^{-3} s^{-1} and T_0 is 282 K for $n = 3$. An advantage of eq.(5) lies in its simple form which enables the integration of eq.(3) to be done analytically with the help of the simple depth profile of temperature to be described in the next section.

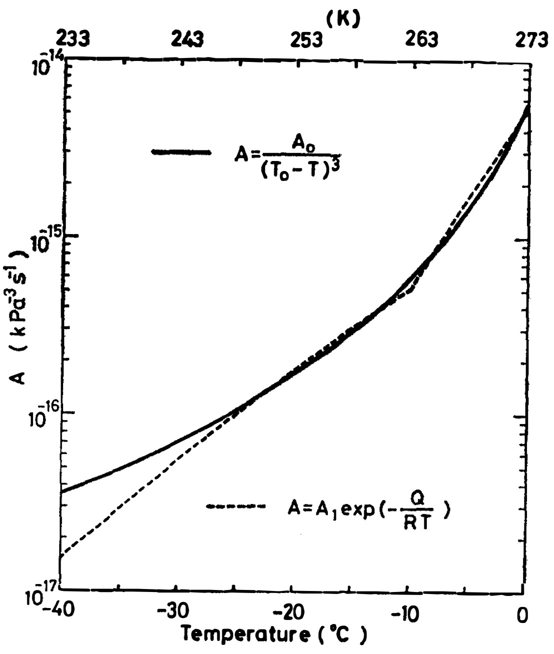

Fig. 2. Temperature dependence of A in a flow law $\dot{\varepsilon}=A\tau^n$ when $n=3$. Full line shows the proposed expression eq.(5), with $A_0=4.2\times10^{-12}$ $kPa^{-3}K^3$ s^{-1}. The dashed line is the Arrhenius relation eq.(4), in which Q is taken to be 139 kJ m^{-1} for $T\geq-10°C$ and 60 kJ mol^{-1} for $T<-10°C$.

4. Vertical Temperature Profile in an Ice Sheet

The vertical temperature profile in an ice sheet in a steady state can be expressed as follows (ROBIN, 1955).

$$T(l)=T_b+a\int_0^l \exp\left(-\gamma\frac{l^2}{2}\right)\mathrm{d}l\,, \tag{6}$$

where T_b is the bottom temperature, a is the temperature gradient at the bottom against l, and $\gamma=bH/\kappa$, in which b is the net balance rate, and κ is the thermal diffusivity of ice. The value for a can be given by

$$a=H\left(\frac{\partial T}{\partial z}\right)_{\text{bottom}}\approx\frac{-HG}{K}\,, \tag{7}$$

in which K is the thermal conductivity of ice and G the geothermal heat. The bottom temperature T_b could be calculated from the surface temperature T_s by

$$T_b=T_s-aF_s(\gamma)\,, \tag{8}$$

in which,

$$F_s(\gamma)=\int_0^1 \exp\left(-\gamma\frac{l^2}{2}\right)\mathrm{d}l\,. \tag{9}$$

Equation (6) was found to represent an actual temperature profile pretty well near the center of an ice sheet, but not for the places where the horizontal ice movement is significant.

The effect of the horizontal ice movement in heat conduction or the horizontal advection effect was taken into account by RADOK *et al.* (1970). The generalized solution can be given by eq.(6) with the same value for a but with a modified expression for T_b as follows:

$$T_b = T_s - aF_s(\gamma) - \frac{WH^2}{\kappa} F_D(\gamma) , \tag{10}$$

where $F_s(\gamma)$ is the same as eq.(9) and F_D is expressed as

$$F_D(\gamma) = \int_0^1 \int_0^l \exp\left\{\frac{\gamma}{2}(l'^2 - l^2)\right\} \mathrm{d}l'\mathrm{d}l . \tag{11}$$

For a steady state condition, W in eq.(10) equals $\lambda u_s \alpha$, where u_s is the surface velocity, λ the lapse rate and α the surface slope.

Since the temperature profile obtained by the column model of RADOK *et al.* is nearly linear in the lower part of the ice sheet, we can assume as a first approximation that the temperature profile in the ice sheet is linear with the depth, that is to say

$$T = T_b + al', \tag{12}$$

where T_b and a are given by eqs.(10) and (7) respectively. In fact, measured temperature profiles in accumulation areas of polar ice sheets exhibit approximately linear relations near the bottom (PATERSON, 1981) as well as the calculated results by RADOK *et al.* (1970) do the same. The possible error caused by this approximation will be discussed in Section 6.

5. Calculation of *I* with a Linear Temperature Profile

Substitution of eq.(5) and eq.(12), into eq.(3) gives

$$I = \int_0^1 \int_0^l \frac{A_0}{(T_a - al')^3} (1 - l')^3 \mathrm{d}l'\mathrm{d}l , \tag{13}$$

where n is taken to be 3, and $T_a = T_0 - T_b$. Equation (13) is integrated to become

$$I = \frac{A_0}{5T_a^3} f\left(\frac{-a}{T_a}\right) , \tag{14}$$

where

$$f(x) = \frac{5}{x^5}\left\{6(1+x)^2 \ln(1+x) - \left(6x + 9x^2 + 2x^3 - \frac{x^4}{2}\right)\right\} , \tag{15}$$

when $a=0$, *i.e.*, the ice sheet is vertically at homogeneous temperature T_b, I becomes $A_0/5T_a^3$, since $f(x)$ is unity for $x=0$. The value of $f(x)$ is regarded as the ratio of the mass fluxes for a given temperature profile designated by T_a and a to that for constant temperature profile of T_b. Numerical values of $f(x)$ as a function of x are illustrated in Fig. 3.

If we calculate values of a from various data of temperature profiles of polar ice sheet (Fig. 10.7, PATERSON, 1981), they are between -10°C and -100°C. The value of a given by eq.(7) is approximately -50°C for 2000 m of H if we adopt appropriate values for G and K. The value of $-a/T_a$ would be, hence, in a range between 0 and 11, as was taken to the range of x on the abscissa of Fig. 3.

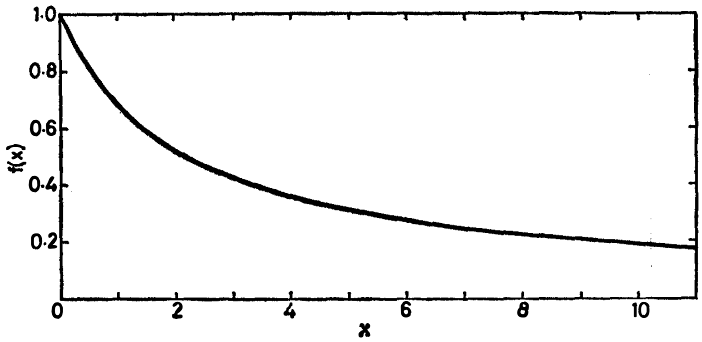

Fig. 3. Graphical expression of f(x) defined by eq.(15).

6. Error in the Estimate of *I*

It has been shown in the preceding sections that the integration for the calculation of the mass flux or the quantity I in eq.(2) can be carried out analytically if we assume a simple form of temperature dependence of the flow law parameter as expressed by eq.(5) and also a linear depth profile of temperature in the ice sheet. Since the real profile deviates from the linear relation, particularly in the part near the surface, we should be cautious about the error of computation when it is to be carried out by the way proposed here. In this section, the error caused by the assumption of linear profile of temperature will be evaluated. Error caused by the approximation of eq.(5) about the temperature dependence of the flow law parameter A is not considered because it is anticipated to be relatively small.

The error is evaluated by comparison with extremes which can be calculated with lower and higher temperature conditions than the real one. Consider four different temperature profiles as shown in Fig. 4. T_1 indicates the linear profile employed in the present calculation of I and this gives the lowest estimate of the temperature profile. T_r (broken line) is the real profile and it should be very close to the theoretical profile given by RADOK *et al.* (1970). Since the Radok's formula is very complex, the profile T_r is substituted by T_2 for simplicity. This is given by eq.(6) (Robin's profile with positive γ for accumulation area) with a value of T_b obtained by eq.(10) of the model of RADOK *et al.* T_2 would be close to T_r, particularly near the bottom. T_3 which gives the highest estimate of the temperature profile is defined by

$$T_3(l)=\begin{cases} T_b+acl & \text{for } 0\leq l\leq l_1 \\ T_b+acl_1=T(l_1) & \text{for } l_1\leq l\leq 1\,. \end{cases} \tag{16}$$

In this equation, c is a correction factor to give a linear approximation for the bottom part ($0\sim l_1$) of T_2 and is given by

$$c=\frac{1}{l_1}\int_0^{l_1}\exp\left(-\frac{\gamma}{2}l^2\right)\mathrm{d}l\,. \tag{17}$$

At the depth l_1, $T_3=T_2$ as can be seen by comparison of eq.(16) and eq.(6). Since T_r is approximated by T_2, it is clear that the real profile T_r is between T_1 and

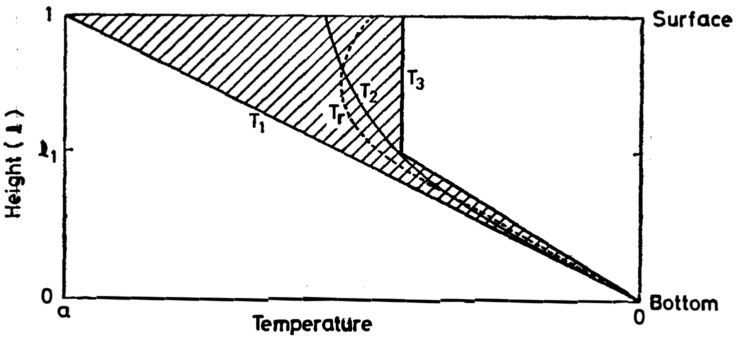

Fig. 4. Schematic temperature profiles, T_1, T_2, T_3, and T_r in an ice sheet of which the depth is expressed by the normalized depth l. T_1 is the linear profile used in the present analysis. T_2 is a modified Robin's steady state profile used for an approximation of the real profile T_r. T_3 is defined by eq.(16).

T_3. In other words, $T_1(l) \leq T_r(l) \leq T_3(l)$ holds at any l.

Since the flow law parameter A increases uniformly with temperature (see Fig. 2), I_1, I_r and I_3, which are the values for I corresponding to T_1, T_r and T_3 respectively, should satisfy the inequalities, $I_1 < I_r < I_3$. The error ΔI caused by the use of T_1 instead of T_r is defined as $\Delta I = (I_r - I_1)/I_r$, and it should be less than $(I_3 - I_1)/I_1$ from the above inequalities. This could be taken as the maximum value of ΔI, that is say

$$\Delta I_{\max} = \frac{I_3 - I_1}{I_1}. \tag{18}$$

Whereas the value of I_1 is given by eq.(14), I_3 is computed by the combination of eq.(14) and eq.(16) as follows:

$$I_3 = \frac{A_0 f\left(\frac{-ca}{T_a}\right)}{5T_a^3} + (1-l_1)^5 \frac{A_0\left\{1 - f\left(\frac{-ca}{T_a - acl_1}\right)\right\}}{5(T_a - acl_1)^3}. \tag{19}$$

$\Delta I_{\max}$ is hence given by

$$\Delta I_{\max} = \frac{1}{f(x)}\left\{f(cx) + (1-l_1)^5 \frac{1 - f\left(\frac{cx}{1+cxl_1}\right)}{(1+cxl_1)^3}\right\} - 1. \tag{20}$$

Determination of the value of l_1 is somewhat arbitrary, but it is so chosen as to make $\Delta I_{\max}$ a minimum. According to eq.(20), values of $\Delta I_{\max}$ were computed as a function of $x = -a/T_a$ and c or γ (see eq.(17)).

$\Delta I_{\max}$ is plotted against x for various γ in Fig. 5. It gradually increases with increasing x for various values of γ at first and then stays almost constant for x larger than about 2. The maximum value on each curve in Fig. 5 was plotted against the value of γ of each. The relationship between them is expressed by a curve in Fig. 6. Figure 5 shows that the $\Delta I_{\max}$ depends primarily upon γ in the range of x over 2 or 3. As mentioned earlier, $\gamma = bH/\kappa$, where b is the net balance rate, H is the ice thickness, and κ is the thermal diffusivity of ice (*ca.* $36\ \mathrm{m}^2 \cdot a^{-1}$, PATERSON, 1981). If we adopt the value $b \approx 0.07\ \mathrm{m \cdot a^{-1}}$ (NARITA and MAENO,

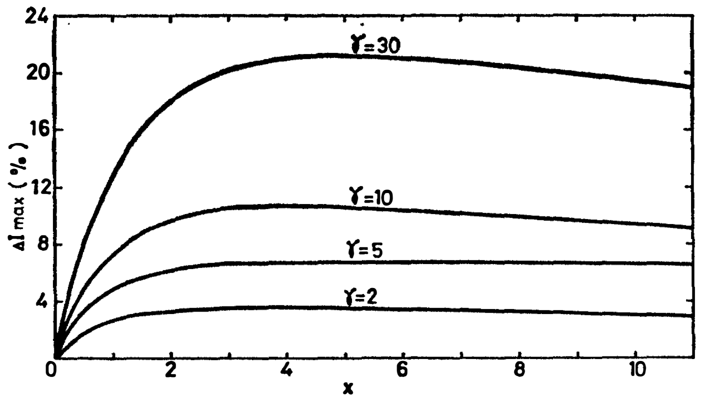

Fig. 5. Estimated error ΔI_{max} as a function of x for various γ.

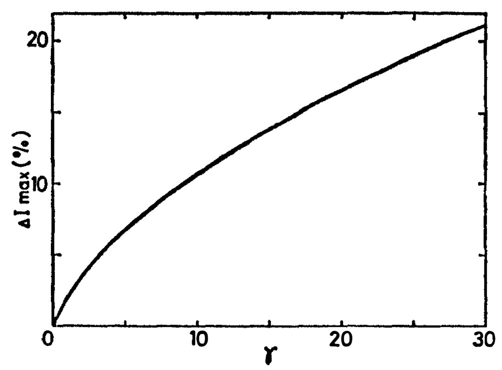

Fig. 6. Relationship between the maximum value of ΔI_{max} on each curve and γ in Fig. 5.

1979) and $H \approx 2000$ m (SHIMIZU *et al.*, 1978), both obtained at Mizuho Station, East Antarctica, the value of γ is approximately 4 and accordingly ΔI_{max} becomes 6%. Over the most of the area of Shirase drainage basin, where $b \leq 0.2$ m·a^{-1} (YAMADA and WAKAHAMA, 1981), and $H \leq 2000$ m (SHIMIZU *et al.*, 1978), γ would be less than about 11. This makes ΔI_{max} 11% from Fig. 6. Therefore we can conclude that the mass flux of ice in this area can be estimated within an accuracy of about 11% by the method proposed in this paper. Actually, the estimate of q_x by this method should be several % less than the real value as can be seen from Fig. 4. Above example of the evaluation of error are only for positive γ, *i.e.* for the accumulation area. However, the same considerations apply for an ablation area, where γ takes negative values.

7. Concluding Remarks

It is shown that the mass flux of ice q_x at any place on an ice sheet can be calculated simply by analytical integration even when the effect of the depth profile of temperature in it is taken into account. For this simplification, it is assumed that the temperature dependence of the flow law parameter obeys eq.(5) and the temperature profile in the ice sheet is linear with depth. The temperature dependent term for the calculation, I in eq.(2) is given by eqs.(13) and (14). The maximum error of I caused by the assumption of the linear temperature profile was estimated to be less than about 11% for conditions in the Shirase drainage basin. Therefore, we conclude that the mass flux of ice can be calculated by eq.(2) probably with an error of less than 11%. The proposed method would reduce the amount of computation considerably in numerical modeling of the ice sheet dynamics, because its essential quantity q_x can be obtained analytically with estimated values of bottom temperature and temperature gradient at the bottom.

Further studies are going on to apply this method for constructing three-dimensional models of ice sheets as the Shirase Glacier basin and adjacent areas in East Queen Maud Land where we are expecting to obtain extensive field data in near future.

Acknowledgments

HITAC M-200 H computer system at the Hokkaido University Computing Center was partly used in preparing the present paper. We thank anonymous referee for valuable advice to make the paper legible. This work was supported by the cooperative research fund from the National Institute of Polar Research.

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(Received June 4, 1982)